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Exponential Convergence of Toom's Probabilistic Cellular Automata

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We obtain cluster expansions for small random perturbations of deterministic Toom's automata in the one-dimensional case. Exponential convergence follows. Analyticity of invariant measures is examined as well as the simplest multi-dimensional case.

KEY WORDS: Toom's condition; probabilistic cellular automata (PCA); cluster expansions; exponential convergence.

1. INTRODUCTION

Processes with local interaction in the "high-temperature" region (i.e., when the interaction is weak) are sufficiently well understood (see review in refs. 1, and 7). One can reasonably assert that the low-temperature region for Gibb's random fields corresponds to small perturbations of deterministic processes with local interaction. One of the deepest results is the proof of stability by Toom[(6)] for his class of deterministic processes. For the simplest processes of this type (e.g., for Stavskaya's model) cluster expansion techniques and, as a result, stability, exponential convergence, and an analytic property were known earlier.[(3–5)] In this paper we solve this problem completely (mainly the one of exponential convergence) for the general Toom's model in the one-dimensional case. For more dimensions only very special cases can be treated with our ideas. The general case now seems to be beyond our reach.

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PCA Formalism

We consider PCAs with memory which describe the stochastic discrete-time evolution of spin variables on the lattice Z^d. We denote the value of the spin at the point $x \in Z^d$ at time $t \in Z$ by $\sigma_{(x,t)} = \pm 1$ and write $\underline{\sigma}_t$ for the configuration at time t; σ_A will denote the configuration on the space-time set $A \subset Z^{d+1}$. We assume that the past of the PCA is fixed: $\sigma_{(x,t)} = +1$ if $t < 0$. The PCA evolves by simultaneous updating of spins. That is, the spin configurations $\underline{\sigma}_{t-1},..., \underline{\sigma}_{t-T}$, $t \geqslant 0$, determine the probabilities $P(\sigma_{(x,t)} \mid \underline{\sigma}_{t-1},..., \underline{\sigma}_{t-T})$, $\sigma_{(x,t)} = \pm 1$ of the spin values at each point x at time t. The natural number T is called the depth of memory. The conditional probability distribution of $\underline{\sigma}_t$ is a product measure given by

$$\prod_{x \in Z^d} P(\sigma_{(x,t)} \mid \underline{\sigma}_{t-1},..., \underline{\sigma}_{t-T}) \tag{1.1}$$

The transition probabilities satisfy the normalization condition

$$\sum_{\sigma_{(x,t)} = \pm 1} P(\sigma_{(x,t)} \mid \underline{\sigma}_{t-1},..., \underline{\sigma}_{t-T}) = 1 \tag{1.2}$$

which is taken into account by writing

$$P(\sigma_{(x,t)} \mid \underline{\sigma}_{t-1},..., \underline{\sigma}_{t-T}) = \tfrac{1}{2}[1 + \sigma_{(x,t)} h_{(x,t)}(\underline{\sigma}_{t-1},..., \underline{\sigma}_{t-T})] \tag{1.3}$$

with $|h_{(x,t)}(\underline{\sigma}_{t-1},..., \underline{\sigma}_{t-T})| \leqslant 1$. We assume that $h_{(x,t)}$ is translation invariant, time homogeneous, and of finite range, that is,

$$h_0(\underline{\sigma}_{-1},..., \underline{\sigma}_{-T}) = h_0(\sigma_U)$$

where $U \subset Z^d \times \{-1,..., -T\}$ is a fixed finite set, which we call the basic set of the origin $\underline{0}$ of the lattice Z^{d+1}. We shall consider nearly deterministic PCAs with

$$h_{\underline{0}} = \Phi_{\underline{0}} \cdot (1 - 2f_0) \tag{1.4}$$

where $|\Phi_{\underline{0}}| = 1$ and $|f_0| \leqslant \varepsilon$ for a small parameter $\varepsilon > 0$. If $f_{\underline{0}} \equiv 0$, we obtain a deterministic PCA with deterministic function $\Phi_{\underline{0}}$. The existence of a small perturbation $f_{\underline{0}}$ implies that for each point $(x, t) \in Z^{d+1}$

$$P(\sigma_{(x,t)} \neq \Phi_{(x,t)}(\underline{\sigma}_{t-1},..., \underline{\sigma}_{t-T})) \leqslant \varepsilon$$

The transition rates (1.1)–(1.4) define a distribution on the space of spin configurations on Z^{d+1}. We shall investigate the limiting behavior of finite-dimensional probabilities $P(\sigma_A)$, $A \subset Z^{d+1}$. Thus we shall say that PCA exponentially converges to the stationary state if the probabilities $P(\sigma_{T^\tau A})$

tend to some limit $\mu(\sigma_A)$ as $\tau \in \mathbb{N}$ tends to infinity and this convergence is exponential: that is, there exist constants $C(A)$ dependent on A and $\gamma < 1$ independent of A such that

$$|\mu(\sigma_A) - P(\sigma_{T^\tau A})| < C(A) \cdot \gamma^\tau$$

Here, by $T^\tau A$ we denote a time shift of A—the set $\{(0, \tau) + a \mid a \in A\}$.

A PCA with the deterministic function Φ_0 is called stable if for any $\delta > 0$ there is an $\varepsilon > 0$ such that for any $z \in Z^{d+1}$ we have $P(\sigma_z = -1) < \delta$ uniformly in f_0, provided $|f_0| < \varepsilon$.

Now consider Toom's criterion of stability for PCAs under consideration given in ref. 6. Toom calls a subset $Q \subset U$ a plus set if $\Phi_0(\sigma_U) = +1$ for any configuration σ_U equal to $+1$ for all $z \in Q$; Q is a minimal plus set if it contains no other plus sets. From now on we shall consider Z^{d+1} as a subset of real space $\mathbb{R}^{d+1}$; for any $A \subset Z^{d+1}$ we denote by Conv(A) the convex hull of A in $\mathbb{R}^{d+1}$. Toom stated his criterion for monotone deterministic functions Φ_0 satisfying $\Phi_0(\sigma'_U) \leqslant \Phi_0(\sigma''_U)$ if $\sigma'_U \leqslant \sigma''_U$, where the last inequality is considered at every point of U. For monotone PCAs satisfying conditions (1.1)–(1.4) Toom's criterion is stated as follows: "A PCA is stable if and only if there is no ray from the origin, intersecting the convex hull of any minimal plus set" (Toom's condition).

We shall prove that for $d = 1$ Toom's condition is sufficient for exponential convergence of the PCA satisfying (1.1)–(1.4) to a stationary state. The requirement of monotonicity may be omitted. For $d \geqslant 1$ we consider one very special case, which is a generalization of the well-known Stavskaya's model.

Certain constructions used in our investigation are rather geometrical, so we need to introduce some, mostly well-known, geometrical notions. These notions are useful both for cases $d = 1$ and $d \geqslant 1$, which is why we shall present them in the introductory section.

For any set $G \subset \mathbb{R}^{d+1}$ by $z + G$ we denote a shift $\{z + g \mid g \in G\}$ of the set G, and we designate $U(z) = z + U$ the basic set of the point $z \in Z^{d+1}$. The set $O = \{\pm v \mid v \in U\} \cup \{v - w \mid v, w \in U\}$ is called a neighborhood of zero and the set $O(z) = z + O$ is the neighborhood of the point z.

The notation $|A|$ will be used for the cardinality of any set $A \subset Z^{d+1}$. For arbitrary $A \subset Z^{d+1}$ we define its height $upA = \max\{t \mid (x, t) \in A\}$ and its top layer $\sup A = \{(x, t) \in A \mid t = upA\}$. By Z_t^{d+1} and $\mathbb{R}_t^{d+1}$ we denote the sets $Z^d \times \{t, t-1,...\}$ and $\mathbb{R}^d \times [0, t]$, respectively. We shall write $R^t(A)$ for the projection of the set $A \subset Z^{d+1}$ into Z_t^{d+1}, so $R^t(A) = A \cap Z_t^{d+1}$; $L^t(A) = \{(x, \tau) \in A \mid \tau = t\}$ is a t-cut of the set A. The set $A \subset Z^{d+1}$ is called connected if for any $u, v \in A$ there exist $z_1 = u, z_2,..., z_m = v$, such that $z_i \in A$ and $z_{i+1} \in O(z_i)$, $i = 1,..., m-1$. For points u and v of this sort we shall write $u \sim^A v$.

The designation σ_A^- will be used for the configuration of -1 on A and σ_A^+ for the configuration of $+1$. For convenience we shall write $P(A_1^+, A_2^+,..., B_1^-, B_2^-,..., \sigma_{C_1}, \sigma_{C_2},...)$ for the probability of the configuration coinciding with $\sigma_{A_i}^+$ on A_i, $\sigma_{B_i}^-$ on B_i, and configuration σ_{C_i} on C_i, $i=1, 2,...$. The number of sets in $P(\cdot)$ is arbitrary. Similarly, a notation

$$P(A_1^+, A_2^+,..., B_1^-, B_2^-,..., \sigma_{C_1}, \sigma_{C_2},..., \mid D_1^+, D_2^+,..., E_1^-, E_2^-,..., \sigma_{F_1}, \sigma_{F_2},...)$$

will be used for the conditional probability of the configuration specified above, given the configuration which agrees with $\sigma_{D_i}^+, \sigma_{E_i}^-, \sigma_{F_i}$, $i=1, 2,...$.

Having described these concepts we now proceed to the meaningful part of our work.

2. THE CASE $d=1$

We shall prove the theorem for $d=1$ under the condition, which is wider than that given by Toom, stated as follows:

- Modified Toom's Condition: There exist two plus sets Q_l and Q_r and a line l passing through the origin and separating these sets in $\mathbb{R}^2$.

Obviously any stable PCA has at least one plus set, because if $\Phi_0(\sigma_U^+) = -1$, the PCA is unstable. Thus we reformulate Toom's condition in a more general form, since we do not require monotonicity of the deterministic function.

Theorem 1. Consider the PCA defined by (1.1)–(1.4) and satisfying the modified Toom's condition. Then for sufficiently small $\varepsilon > 0$ the PCA converges exponentially to the stationary state.

We shall give the proof for the case $T=1$ only. The case $T>1$ is similar, but has rather annoying technicalities (see Berezner[8]). Besides, our method allow us to consider a few modifications of the considered models. For example, we can allow the perturbation f_0 from (1.4) not to be bounded by ε for configurations σ_U where $\Phi_0(\sigma_U) = -1$. We can apply our constructions for more values of $\sigma_{(x,t)}$.[8]

Remark 1. To prove the theorem, we have to prove the exponential convergence of probability $P(\sigma_{T^\tau A}^-)$, $\tau \to \infty$. Our goal is to obtain the exponentially convergent (uniformly in τ) series $\sum_n C_n^\tau(A)$ for the probability $P(\sigma_{T^\tau A}^-)$, where the coefficients of the two series $\sum C_n^{\tau_1}(A)$ and $\sum C_n^{\tau_2}(A)$ can differ only for $n \geqslant \min(\tau_1, \tau_2)$. This will immediately imply the exponential convergence of $P(\sigma_{T^\tau A}^-)$ as $\tau \to \infty$.

For this purpose we shall introduce a special construction, which we shall call the cluster expansion.

The Construction of the Cluster

For convenience we assume that the basic set U of the origin is the set of points $(-v, -1),..., (v, -1)$, $v \in \mathbb{N}$, adding points of fictitious dependence.

Consider arbitrary finite sets $A, F \subset Z^2$ such that $upF < upA = t$, and fix an arbitrary configuration $\underline{\sigma}$ on Z_t^2 coinciding with σ_A^- on A and some fixed configuration σ_F on F.

Remark 2. The necessity of considering set F with fixed configuration on it will become clear later. Such a set will represent a set of points where the configuration will be fixed as a result of using the multistep cluster expansion.

For any such configuration we obtain a partition of Z_t^2 into the maximal connected sets $\{G_\omega^-\}$, $\omega \in \Omega^+$, where $\underline{\sigma}$ is equal to $+1$. Let us choose sets $G_1^-,..., G_m^-$ satisfying

$$\sup A \subset \bigcup_{i=1}^{m} G_i^- \qquad \text{and} \qquad \sup A \cap G_i \neq \varnothing, \qquad i = 1,..., m \tag{2.1}$$

We shall call such sets carrier sets of the cluster. To any carrier set we assign a component of the cluster in the same way. So, to avoid unnecessary designations, we shall consider in detail the case of the unique carrier set G_1^-, which we shall denote by omitting indices. The case of the multicomponent carrier will then be obtain by simply taking the union of corresponding components of the cluster.

Taking the set G, we shall assign to it a set $\Gamma \subset Z_t^2$, which we shall call the cluster, a corresponding oriented contour $\bar{\Gamma}$ in $\mathbb{R}_t^2$, and the set $\partial\Gamma \subset Z_t^2$, called the boundary of Γ.

We shall consider only such configurations $\underline{\sigma}$ on Z_t^2 for which carrier sets are finite, because it is easy to prove that the probability of the infinite carrier set of a finite set A is zero. To do this, let us take a sequence of lines $l_k = l + (0, k(2v+1))$ in $\mathbb{R}^2$, $k \in Z$ (remember that l is a separating line from the modified Toom condition). By J_k we denote a set $\{(x, \tau) \in Z_t^2, \tau \geqslant 0 |$ either (x, τ) lies in the band between l_k and l_{k+1} or between l_{-k} and $l_{-k-1}\}$. As a result of the choice of J_k and the properties of separating lines we have $P(J_{2k}^+) \geqslant (1-\varepsilon)^{|J_{2k}|}$, and the configurations $\sigma_{J_{2k}}^+$, $k \in \mathbb{N}$ are independent. As $\sum_{k=1}^{\infty} P(J_{2k}^+) = \infty$ we obtain that with the probability one, there exist two bands of $+1$ points, separating any set inside from infinity, which implies that any carrier set of the cluster is finite with probability one. Now denote

$$\bar{G} = \bigcup_{u, v \in G : u \in O(v)} [u, v] \subset \mathbb{R}^2$$

where any $[u, v]$ is a segment from $\mathbb{R}^2$. There exists a closed, oriented anticlockwise contour $\bar{\Gamma}$ formed by the ordered set of segments $[z_0, z_1]$, $[z_1, z_2]$,..., $[z_{n-1}, z_n]$, $[z_n, z_0]$ oriented from the first to the second point and satisfying the following conditions:

(a) $z_{i+1} \in O(z_i)$, $[z_i, z_{i+1}] \cap G = \{z_i, z_{i+1}\}$, $i = 1,..., n$.

(b) $O(z_i) \cap \bar{G} \cap Ang(z_{i-1}, z_i, z_{i+1}) = \varnothing$, $i = 1,..., n$, where $Ang(u, z, v)$ is the angle in $\mathbb{R}^2$ formed by the rays $[z, u)$ and $[z, v)$, turning from the first to the second ray anticlockwise.

(c) The set $\bar{G} \backslash \bar{\Gamma}$ belongs to the internal domain $\hat{\Gamma} \subset \mathbb{R}_t^2$ of the contour $\bar{\Gamma}$, which lies to the left of the contour $\bar{\Gamma}$, while traversing $\bar{\Gamma}$ anticlockwise.

(d) There does not exist any closed subcontour formed by the segments of $\bar{\Gamma}$ and oriented clockwise.

The conditions (a)–(d), and especially the condition (b), are very constructive, and one can easily check the existence of the contour mentioned by straightforward construction of the contour. To do this, one can start from the point $z_0 \in \sup G$ with the minimal x coordinate and then determine uniquely at every step the next segment of $\bar{\Gamma}$ in keeping with the conditions in (a)–(b). Using loose but descriptive geometric terminology, we can say that $\bar{\Gamma}$ is an external oriented geometrical boundary of the set $\bar{G}$.

The cluster Γ corresponding to the set G (and to the configuration $\underline{\sigma}$ coinciding with σ_A^- on A and σ_F on F) is the set $\Gamma = \bar{\Gamma} \cap G$. The set

$$\partial\Gamma = \left\{ \bigcup_{z \in \Gamma} O(z) \right\} \Big\backslash (\hat{\Gamma} \cup \Gamma) \cap Z_t^2 \tag{2.2}$$

is called the boundary of the cluster Γ.

In the case where the carrier of the cluster consists of few components G_1^-,..., G_m^- let us assign to every component G_i^- the corresponding component Γ^i of the cluster, contour $\bar{\Gamma}^i$, and boundary $\partial\Gamma^i$ and internal domain $\hat{\Gamma}^i$. Then we define $\Gamma = \bigcup \Gamma^i$; $\hat{\Gamma} = \bigcup \hat{\Gamma}^i$; $\partial\Gamma = \bigcup \partial\Gamma^i$; $\bar{\Gamma} = \bigcup \bar{\Gamma}^i$; $i = 1,..., m$.

Thus to any configuration $\underline{\sigma}$ coinciding with σ_A^- on A and σ_F on F we assign the cluster Γ, which we shall call the cluster Γ with kernel A and fixed configuration σ_F.

The immediate consequence of the construction of the cluster is the fact that $\underline{\sigma}$ coincides with σ_Γ^- on Γ and $\sigma_{\partial\Gamma}^+$ on $\partial\Gamma^+$. Moreover, if any other configuration σ agrees with σ_A^-, σ_Γ^-, $\sigma_{\partial\Gamma}^+$, and σ_F, then its cluster is the same.

Now we shall formulate one important property of clusters, which will be proved at the end of this section.

We shall call a point $z \in \Gamma$ an error point if at least one of the plus sets $z+Q_r$ or $z+Q_l$ belongs to $\partial\Gamma$; this will imply $\Phi_z(\sigma_{U(z)}) = +1$. The set of error points is denoted by $\varepsilon(\Gamma)$.

Lemma 1. There exists sufficiently small $\alpha > 0$ such that for any Γ

$$|\varepsilon(\Gamma)| > \alpha \cdot |\Gamma| \tag{2.3}$$

This lemma will be proved at the end of this section.

Now we are ready to give the cluster expansion for the probability $P(\sigma_{A_0}^-)$ of the configuration on an arbitrary finite set $A_0 \subset Z_t^2$ (we introduce "0" to show that we are at the starting point of our expansion). Taking $t = upA_0$, we consider all configurations $\underline{\sigma}$ in Z_t^2 coinciding with $\sigma_{A_0}^-$ on A_0, and assign to each such configuration the cluster $\Gamma_0 = \Gamma_0(\underline{\sigma})$ with kernel A_0 and corresponding sets $\partial\Gamma_0, \Gamma_0, \Gamma_0$. Taking into account the relation between $\underline{\sigma}$ and the cluster Γ_0, we can give the expansion

$$P(\sigma_{A_0}^-) = \sum_{\Gamma_0} \sum_{\underline{\sigma}:\Gamma_0(\underline{\sigma}) = \Gamma_0} P(\sigma) = \sum_{\Gamma_0} P(A_0^-, \Gamma_0^-, \partial\Gamma_0^+) \tag{2.4}$$

where the sum is over all possible clusters Γ_0.

Let us consider the set D_0 from $Z \times \{0, 1, 2,...\}$ defined by

$$D_0 = \left\{ \bigcup_{z \in \sup \partial\Gamma_0} U(z) \right\} \Big\backslash (A_0 \cup \partial\Gamma_0 \cup \hat{\Gamma}_0 \cup \Gamma_0) \tag{2.5}$$

So the points of D_0 belong to the basic set of sup $\partial\Gamma_0$ and lie outside the interior of $\bar{\Gamma}_0$ and the values of the configuration $\underline{\sigma}$ at these points are not fixed by the algorithm for constructing the cluster at the previous step. We can rewrite the probabilities on the right-hand side of (2.4) as

$$P(A_0^-, \Gamma_0^-, \partial\Gamma_0^+) = \sum_{\sigma_{D_0}} P(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, \sigma_{D_0}) \tag{2.6}$$

We shall write $P_t(A^-, B^+, \sigma_C,...)$ for the probability of the corresponding configuration on the projection of the sets in the parentheses in Z_t^2.

Then the probability on the right side of (2.6) can be written as

$$\begin{aligned} &P(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, \sigma_{D_0}) \\ &\quad = P_t(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, \sigma_{D_0}) \\ &\quad = P_{t-1}(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, \sigma_{D_0}) \\ &\qquad \times P(L^t(A_0 \cup \Gamma_0)^-, L^t(\partial\Gamma_0)^+ \mid R^{t-1}(A_0 \cup \Gamma_0)^-, R^{t-1}(\partial\Gamma_0)^+, \sigma_{D_0}) \end{aligned} \tag{2.7}$$

The second probability on the right-hand side is the conditional probability of the configuration on the t-cut $L^t(A_0 \cup \Gamma_0 \cup \partial\Gamma_0)$ under the condition of a fixed configuration on the projection of the set $A_0 \cup \Gamma_0 \cup \partial\Gamma_0 \cup D_0$ in Z^2_{t-1}.

Taking into account that any configuration σ_{D_0} can be written as $\sigma^-_{B_0} \cup \sigma^+_{D_0 \backslash B_0}$ for some $B_0 \subset D_0$, we write

$$P(\sigma_{D_0}) = P(B_0^- \cup (D_0 \backslash B_0))^+ = \sum_{W_0 : B_0 \subset W_0 \subset D_0} (-1)^{|W_0 \backslash B_0|} \cdot P(W_0^-) \quad (2.8)$$

Using the expression (2.7) and an argument similar to what is used in deriving formula (2.8), we can write expansion (2.6) as

$$P(A_0^-, \Gamma_0^-, \partial\Gamma_0^+) = \sum_{W_0 \subset D_0} Q^t_{\text{cond}}(A_0, \Gamma_0, \partial\Gamma_0, W_0) \cdot P_{t-1}(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, W_0^-) \quad (2.9)$$

with

$$\begin{aligned} Q^t_{\text{cond}}(A_0, \Gamma_0, \partial\Gamma_0, W_0) &= \sum_{B_0 \subset W_0} (-1)^{|W_0 \backslash B_0|} \\ &\quad \times P(L^t(A_0 \cup \Gamma_0)^-, L^t(\partial\Gamma_0)^+ \mid \\ &\qquad R^{t-1}(A_0 \cup \Gamma_0 \cup B_0)^-, R^{t-1}(\partial\Gamma_0 \cup D_0 \backslash B_0)^+) \end{aligned} \quad (2.10)$$

The 4-tuple of sets $A_0, \Gamma_0, \partial\Gamma_0, W_0$ is denoted by H_0. A very important consequence of the definition (2.5) of the set D_0 and property (2.2) of the boundary $\partial\Gamma_0$ is the fact that conditional probabilities in (2.10) do not depend on the time shift T^τ, $\tau \in \mathbb{N}$, of all constructed sets. Thus if we apply our construction to the probabilities

$$P(\sigma_{T^{\tau_i} A_0}), \qquad i = 1, 2$$

we obtain

$$Q^{t+\tau_1}_{\text{cond}}(T^{\tau_1} H_0) = Q^{t+\tau_2}_{\text{cond}}(T^{\tau_2} H_0) \quad (2.11)$$

This fact together with the property (2.3) and with the expansion (2.9) is the cornerstone of the expansion mentioned in Remark 1. We need only show how the probabilities on the right-hand side of (2.9) can be expressed through the probabilities in Z^2_{t-2} in a similar way. Considering the probability $P_{t-1}(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, W_0^-)$ in (2.9), we determine $A_1 = W_0 \cup (A_0 \backslash (\Gamma_0 \cup \hat{\Gamma}_0))$ and take an arbitrary configuration $\underline{\sigma}$ on Z^2_{t-1} coinciding with $\sigma^-_{A_1}$ on A_1 and σ_{F_1} on the set $F_1 = R^{t-1}(H_0) \backslash A_1$, where σ_{F_1}

has values fixed on the set F_1 from the previous step of our algorithm. Here we write $R^{t-1}(H_0)$ for the union of the sets $R^{t-1}(A_0)$, $R^{t-1}(\Gamma_0)$, $R^{t-1}(\partial\Gamma_0)$, $R^{t-1}(W_0)$. For each such configuration $\underline{\sigma}$ on Z^2_{t-1} with the fixed configuration on $A_1 \cup B_1$ we construct cluster Γ_1 with the kernel A_1 (see the section, "The Construction of the Cluster"). Thus

$$P_{t-1}(A_0^-, \Gamma_0^-, \partial\Gamma_0^+, W_0^-) = \sum_{\Gamma_1} P_{t-1}(\sigma_{F_1}, A_1^-, \Gamma_1^-, \partial\Gamma_1^+) \tag{2.12}$$

Defining D_1 in the same way as we have defined D_0 [see general definition (2.13)], we can get the expansion for the probability on the left-hand side of (2.12) through the probabilities in Z^2_{t-2}. Thus the expansion of the type of (2.9) can be iterated many times. The difference is only in the growing number of points where the value of the configuration is fixed at the previous stages. Remember that at every stage we move down the t axis by a unit step. Assuming that we have constructed the chain of 4-tuples $H_k = \{A_0, \Gamma_0, \partial\Gamma_0, W_0\},..., \{A_k, \Gamma_k, \partial\Gamma_k, W_k\}$, let us determine the 4-tuple of the $(k+1)$th step. We denote $A_{k+1} = A_k \backslash (\Gamma_k \cup \hat{\Gamma}_k) \cup W_k$ and $F_{k+1} = R^{t-k-1}(H_k) \backslash A_{k+1}$. The cluster Γ_{k+1} with the kernel A_{k+1} is then constructed for the configuration $\underline{\sigma}$ in Z^2_{t-k-1} with fixed values at the points of $R^{t-k-1}(H_k)$. The set D_{k+1} is the set from $Z \times \{0, 1, 2,...\}$ defined by

$$D_{k+1} = U\left(\bigcup_{j=0}^{k+1} (\partial\Gamma_j \cap Z^2_{t-k-1})\right) \Big\backslash (A_{k+1} \cup \partial\Gamma_0 \cup \cdots \cup \partial\Gamma_{k+1} \cup \hat{\Gamma}_0 \cup \cdots \cup \hat{\Gamma}_{k+1}) \tag{2.13}$$

where $U(G) = \bigcup_{z \in G} U(z)$ is the basic set of the set G.

The expansion (2.9) of the $(k+1)$th step is then

$$\begin{aligned} P_{t-k-1}&(A_0^-, \Gamma_0^-, \partial\Gamma_0^+,..., A_{k+1}^-, \Gamma_{k+1}^-, \partial\Gamma_{k+1}^+) \\ &= P_{t-k-1}(\sigma_{F_{k+1}}, A_{k+1}^-, \Gamma_{k+1}^-, \partial\Gamma_{k+1}^+) \\ &= \sum_{\Gamma_{k+1}, W_{k+1} \subset D_{k+1}} Q_{\text{cond}}^{t-k-1}(H_{k+1}) \\ &\quad \times P_{t-k-2}(\sigma_{F_{k+1}}, A_{k+1}^-, \Gamma_{k+1}^-, \partial\Gamma_{k+1}^+, W_{k+1}^-) \end{aligned} \tag{2.14}$$

where

$$Q_{\text{cond}}^{t-m}(H_m) = \sum_{B_m \subset W_m} (-1)^{|W_m \backslash B_m|} \times P\left(L^{t-m}(A_m \cup \Gamma_m)^-, L^{t-m}\left(\bigcup_{i=1}^{m} \partial\Gamma_i\right)^+ \,\Big|\, R^{t-m-1}(E_m)^-, R^{t-m-1}(\sigma_{F_m}), (D_m \backslash B_m)^+\right)$$

By E_m we denote in this formula the set $A_m \cup \Gamma_m \cup B_m$, and m is equal to $k+1$. At the step j when for the first time

$$\varnothing = A_{j+1} = W_j = R^{t-j}(\partial \Gamma_0 \cup \cdots \cup \partial \Gamma_j) \cap Z \times \{0, 1, 2, ...\} \quad (2.15)$$

we say that the chain H_j is truncated. The contribution $S(H_j)$ of the truncated chain H_j in the expansion of probability $P(\sigma^-_{A_0})$ is

$$S(H_j) = \prod_{k=0}^{j} Q^{t-k}_{\text{cond}}(H_k) \quad (2.16)$$

where $Q^{t-j}_{\text{cond}}(H_j)$ is equal to 1.

Thus we can write the complete expansion for $P(\sigma^-_{A_0})$ as the sum of contributions over all truncated chains H_j,

$$P(A_0^-) = \sum_{j=0}^{t} \sum_{H_j} S(H_j) \quad (2.17)$$

To each truncated chain H_j we assign the value $|\Gamma| = |\Gamma_0| + \cdots + |\Gamma_j|$. Then we have

$$P(A_0^-) = \sum_{n=1}^{\infty} \sum_{H_j : |\Gamma| = n} S(H_j) = \sum_{n=1}^{\infty} C_n(A) \quad (2.18)$$

Let us estimate the terms of this series. Consider H_j with $|\Gamma| = n$. It follows from (2.3) that

$$|S(H_j)| = \left| \prod_{k=0}^{j} Q^{t-k}_{\text{cond}}(H_k) \right| < 2^{|D|} \cdot \varepsilon^{\alpha n} \quad (2.19)$$

where $|D| = |D_0| + \cdots + |D_j|$. Now we shall show that the number of H_j with $|\Gamma| = n$ is bounded by C^n for some $C > 0$. We have four obvious estimates. The first one is

$$|\{\Gamma_i \mid |\Gamma_i| = k\}| < d_1^k \qquad \text{for} \quad d_1 > 0 \quad (2.20)$$

the second is

$$|\partial \Gamma| = |\partial \Gamma_0| + \cdots + |\partial \Gamma_j| < d_2 \cdot |\Gamma| \qquad \text{for some} \quad d_2 > 0 \quad (2.21)$$

the third is

$$|D| < d_3 \cdot |\partial \Gamma| \qquad \text{for some} \quad d_3 > 0 \quad (2.22)$$

and the fourth claims that the number $\|W\|$ of choices of the sets $W_0, ..., W_j$ from the sets $D_0, ..., D_j$ is bounded by

$$\|W\| \leqslant 2^{|W_0| + \cdots + |W_j|} \leqslant 2^{|D|} \quad (2.23)$$

From these estimates we obtain

$$|\{H_j \mid |\Gamma| = n\}| \leqslant C^n \qquad \text{for some} \quad C > 0 \tag{2.24}$$

Combining (2.19)–(2.24), we get that for some constant $\Psi > 0$

$$|C_n(A)| < (\Psi \cdot \varepsilon^{\alpha})^n \tag{2.25}$$

This proves that for sufficiently small $\varepsilon > 0$ the series (2.18) converges exponentially. Considering a similar expansion for $P(\sigma_{\overline{T^{\tau}A}})$

$$P(\sigma_{\overline{T^{\tau}A}}) = \sum_{n=1}^{\infty} C_n^{\tau}(A) \tag{2.26}$$

we note that the coefficients $C_n^{\tau}(A)$ and $C_n(A)$ for $n < t$ are the sums over congruent sets $T^{\tau}H_j$ and H_j. Taking into account the equality (2.11), we find that these coefficients coincide for $n < t$. Having proved this fact, we refer the reader to Remark 1, which completes the proof of the theorem.

Proof of Lemma 1. Obviously, it is enough to prove (2.3) in the case when Γ consists of one component. Let n_+ be the number of plus vectors in the oriented contour $\bar{\Gamma} = [z_0, z_1], ..., [z_{n-1}, z_n], [z_n, z_0]$, where the oriented segment $\overrightarrow{z_i z_{i+1}}$ is called a plus vector if z_{i+1} lies to the right of the line $z_i + l$. Consider an arbitrary plus vector $\overrightarrow{z_i z_{i+1}}$. We claim that either z_i or z_{i+1} is an error point. Certainly, if $\text{pr}_{\vec{e}} \overrightarrow{z_i z_{i+1}} > 0$ [where $\vec{e}$ is a unit vector $\overrightarrow{0e}$, $e = (0, 1)$, and $\text{pr}_{\vec{a}} \vec{b}$ is the orthogonal projection of the vector $\vec{b}$ on the vector $\vec{a}$], then $\text{pr}_{\vec{e}} \overrightarrow{z_{i+1} z_{i+2}} \geqslant 0$ (otherwise z_i and z_{i+2} will be connected by the segment $[z_i, z_{i+2}]$, but this contradicts the construction of the contour $\bar{\Gamma}$). This fact implies that $z_{i+1} + Q_r(z_{i+1}) \subset \partial\Gamma$ and, correspondingly, $z_{i+1} \in \varepsilon(\Gamma)$. If $\text{pr}_{\vec{e}} \overrightarrow{z_i z_{i+1}} < 0$, then by the same argument $\text{pr}_{\vec{e}} \overrightarrow{z_{i-1} z_i} \leqslant 0$ and $z_i \in \varepsilon(\Gamma)$. If $\text{pr}_{\vec{e}} \overrightarrow{z_i z_{i+1}} = 0$, then either z_{i-1} lies to the left of $z_i + l$ [and this implies $z_i \in \varepsilon(\Gamma)$] or z_{i+2} lies to the right of $z_{i+1} + l$— which implies that $z_{i+1} \in \varepsilon(\Gamma)$. Thus we assign the error point to an arbitrary plus vector.

Let us note that each error point corresponds to no more than $2|O|$ plus vectors, so that

$$|\varepsilon(\Gamma)| > \alpha_1 \cdot n_+, \qquad \alpha_1 = 1/2|O| \tag{2.27}$$

As the contour $\bar{\Gamma}$ is closed, there exists a constant $\alpha_2 > 0$ such that

$$n_+ > \alpha_2(n + 1 - n_+) \tag{2.28}$$

Since $|\bar{\Gamma}| = n + 1 > \alpha_3 |\Gamma|$ for some $\alpha_3 > 0$, then summing (2.27) and (2.28), we obtain

$$|\varepsilon(\Gamma)| > \alpha |\Gamma| \qquad \text{for some} \quad \alpha > 0$$

which ends the proof of Lemma 1.

3. THE CASE $d=1$. ANALYTICITY FOR THE GENERALIZED STAVSKAYA'S MODEL

In this section we consider PCA on Z^d, $d \geqslant 1$, with deterministic function $\Phi_{\underline{0}}$ from (1.4) defined by

$$\Phi_{\underline{0}}(\sigma_U) = \begin{cases} -1 & \sigma_U = \sigma_U^- \\ +1 & \sigma_U \neq \sigma_U^- \end{cases}$$

and perturbation $f_{\underline{0}}$ vanishing on σ_U^-,

$$f_{\underline{0}} = \begin{cases} \varepsilon & \sigma_U \neq \sigma_U^- \\ 0 & \sigma_U = \sigma_U^- \end{cases}$$

For convenience we shall number points (vectors) of the set U in an arbitrary order: $U = \{u_1,..., u_s\}$. We remark that given $d=1$ and basic set $U = \{(0, -1), (-1, -1)\}$, we obtain the well-known Stavaskaya's model, which is why we decided to call the model defined by (1.1)–(1.6) a generalized Stavskaya's model (GSM). As every point of the basic set is a plus set for GSM, Toom's criterion for the GSM takes a very simple form: "GSM is stable iff vectors $u_1,..., u_s$ are not collinear" (we shall call this Toom's condition for the GSM). The main result of this section is in the following theorem.

Theorem 2. For sufficiently small $\varepsilon > 0$ the GSM under Toom's condition converges exponentially to the limiting stationary state. Limiting finite-dimensional distributions are analytic functions of ε in a neighborhood of zero.

Our study of the GSM of such a type is based on ideas and arguments similar to those of Section 2. In the same way as in Section 2, we start studying the probability $P(\sigma_A^-)$, $A \subset Z_t^{d+1}$ by introducing the cluster construction.

The Construction of the Cluster

Consider an arbitrary finite set $A \subset Z^{d+1}$, and fix an arbitrary configuration $\underline{\sigma}$ on Z_t^{d+1}, coinciding with σ_A^- on A. The set $\Gamma \subset Z_t^{d+1}$ is called a cluster corresponding to the configuration $\underline{\sigma}$ [we will denote it as $\Gamma = \Gamma(\sigma)$] if it satisfies the following conditions:

(i) $\sup A \subset \Gamma$; each connected component of Γ contains at least one point from $\sup A$

(ii) $z \in Z_t^{d+1}$, $U(z) \subset \Gamma \Rightarrow z \in \Gamma$ (3.1)

(iii) $A \cap \partial\Gamma = \varnothing$, where $\partial\Gamma$ is a boundary set $\{z \in Z_t^{d+1} \backslash \Gamma \mid O(z) \cap \Gamma \neq \varnothing\}$

(iv) the configuration $\underline{\sigma}$ agrees with σ_Γ^- on Γ and $\sigma_{\partial\Gamma}^+$ on $\partial\Gamma$

Thus to any configuration $\underline{\sigma}$ coinciding with σ_A^- on A we assign the cluster Γ, which we shall call the cluster Γ with the kernel A. The immediate consequence of the construction of the cluster is in the fact that for any other configuration σ that agrees with σ_A^-, σ_Γ^-, and $\sigma_{\partial\Gamma}^+$ its cluster is the same.

We introduce the set $\Delta\Gamma$ of adjacent points in Γ by

$$\Delta\Gamma = \{z \in \Gamma \mid O(z) \cap \partial\Gamma \neq \varnothing\}$$

and the set $\Gamma \backslash \Delta\Gamma$ of internal points.

The point $z \in \Gamma$ is called an error point if $U(z)$ does not belong to Γ; this will imply $\Phi_z(\sigma_{U(z)}) = +1$. This definition agrees with the definition of Section 2. The set of error points is denoted by $\varepsilon(\Gamma)$.

Using the relations between configurations and its clusters, we can give the expansion

$$P(\sigma_A^-) = \sum_\Gamma \sum_{\underline{\sigma}:\Gamma(\underline{\sigma}) = \Gamma} P(\sigma) = \sum_\Gamma P(A^-, \Gamma^-, \partial\Gamma^+) \tag{3.2}$$

where the sum is over all possible (including infinite) clusters Γ_0.

To obtain the situation mentioned in the remark, we need to prove that the number of finite clusters grows at worst exponentially in the cardinality of $\Delta\Gamma$, i.e., for some $C > 0$

$$|\{\Gamma \mid |\Delta\Gamma| = n\}| \leqslant C^n \tag{3.3}$$

and that the estimate

$$|\varepsilon(\Gamma)| > \alpha\, |\Delta\Gamma| \tag{3.4}$$

holds for all finite clusters for some $\alpha > 0$. These estimates will also be used to show that the probability of having an infinite cluster is equal to zero.

We start the proof of (3.3) by showing that the adjacent set of each component of Γ is connected.

Lemma 2. For any connected set Γ satisfying condition (ii) the adjacent set $\Delta\Gamma$ is connected.

Proof. We use induction on the cardinality of Γ. If $|\Gamma| = 1$, then $\Delta\Gamma$ is connected. Assume that for any set of the type considered and of cardinality less than or equal to n its adjacent set is connected. Consider the arbitrary connected set Γ of cardinality $n+1$, satisfying condition (ii). We shall prove that $\Delta\Gamma$ is connected.

Choose a point $z_0 \in \Gamma$ with a minimal t coordinate. The set $\Gamma \backslash z_0$ can be represented as $\Gamma \backslash z_0 = \bigcup_{i=1}^m \Gamma_i$, where m is no greater than $2d$ and each component Γ_i is connected and satisfies condition (ii). According to our inductive assumption all the $\Delta\Gamma_i$ are connected. So, if $\Delta\Gamma = z_0 \bigcup_{i=1}^m \Delta\Gamma_i$, then $\Delta\Gamma$ is connected, because z_0 is connected with all the $\Delta\Gamma_i$. In the other case the addition of z_0 to the set $\bigcup_{i=1}^m \Gamma_i$ will change the status of some points in $\Delta\Gamma_i$—they will become internal instead of adjacent (we shall call such points changed points). It is easy to see that owing to the t-coordinate minimal property of z_0 changed points are of the type $z = z_0 - u_\alpha$ for some $u_\alpha \in U$. Cases $z = z_0 + u_\alpha$ and $z = z_0 - u_\alpha + u_\beta$ can easily be seen to be impossible. Now we shall prove that for any adjacent point $y \in \Delta\Gamma$ connected with the changed point z the point y is connected with z_0 in $\Delta\Gamma$. Thus the change of the status of the changed points does not violate the connectivity of $\Delta\Gamma$ induced by $\Delta\Gamma_i$, $i = 1,..., m$, due to the contribution of z_0. We shall consider all possible representations for the adjacent point y connected to the changed point $z = z_0 - u_\alpha$. The following cases are possible:

(a) $y = z + u_\beta$, $u_\beta \in U$.

Then $y = z_0 - u_\alpha + u_\beta$, so $y \sim^{\Delta\Gamma} z_0$.

(b) $y = z - u_\beta + u_\gamma$, $u_\beta, u_\gamma \in U$.

In accordance with the minimal t-coordinate property of z_0, the point $z_0 - u_\alpha + u_\gamma = z + u_\gamma = y + u_\beta$ belongs to $\Delta\Gamma$, hence $y \sim^{\Delta\Gamma} y + u_\beta = z + u_\gamma$, and by (a), $z + u_\gamma \sim^{\Delta\Gamma} z_0$.

(c) $y = z - u_\beta$, $u_\beta \in U$.

If $y - u_q + u_p \in \partial\Gamma$ for some $u_q, u_p \in U$, then we have $z - u_q + u_p \in \Gamma$ and $z - u_\beta + u_p = y + u_p \in \Gamma$. From the fact that

$$y - u_q + u_p = (z - u_\beta + u_p) - u_q = (z - u_q + u_p) - u_\beta$$

belongs to $\partial\Gamma$ we find that $z - u_\beta + u_p$ and $z - u_q + u_p$ belong to $\Delta\Gamma$. Thus $y \sim^{\Delta\Gamma} z - u_q + u_p$ and by (b), $z - u_q + u_p \sim^{\Delta\Gamma} z_0$. So, if $y - u_q + u_p \in \partial\Gamma$ for some $u_q, u_p \in \partial\Gamma$, then $y \sim^{\Delta\Gamma} z_0$. Consider a case when for any $u_q, u_p \in U$, $y - u_q + u_p \notin \partial\Gamma$. Then $y - u_q + u_p \in \Gamma$ for all $u_q, u_p \in U$ and $y - u_\tau \in \partial\Gamma$ for some $u_\tau \in U$. But then $(y - u_\tau) + u_p \in \Gamma$ for any $u_p \in U$, so $y - u_\tau$ must belong to Γ according to the condition (ii) from (3.1). This contradiction completes the proof of the lemma, as we have checked all the possible cases.

Now, to finish with the proof of (3.3), let us fix a point $z \in A$. According to the Toom's condition for GSM, we have at least two vectors, say u_1 and u_2, which are not collinear. For some $i(\Gamma) \in \{0, 1, 2,...\}$ a point $z_1 = z + i(\Gamma) \cdot u_1 \in \Delta\Gamma$, and for some $j(\Gamma) \in \{0, 1, 2,...\}$ a point $z_2 = z + j(\Gamma) \cdot u_2 \in \Delta\Gamma$. As the set $\Delta\Gamma$ is connected, there exist a constant β such that for all clusters Γ we will have $i(\Gamma) + j(\Gamma) < \beta\,|\Delta\Gamma|$. That means that there exists some $C>0$ such that we have no more than C^n different adjacent sets of cardinality n for clusters containing the point z. Combining this with the fact that the adjacent set of the cluster completely determines the cluster, we finish the proof of the estimate (3.3).

Now we prove the estimate (3.4).

Lemma 3. There exists sufficiently small $\alpha>0$ such that for all finite clusters Γ

$$|\varepsilon(\Gamma)| > \alpha\,|\Delta\Gamma|$$

Proof. Taking an arbitrary point $z\in\Delta\Gamma$, we shall assign to it an error point. The number of points assigned to each error point will be bounded by the same constant. Consider a parallelepiped

$$P(z) = z \cup \left\{ z + \sum_{i=1}^{j} \pi_q(u_i) \,\middle|\, j, q = 1,..., s \right\}$$

where $\pi_q(u_i) = u_{i+q(\text{mod } s)}$. If $P(z)$ contains an error point, we shall assign it to z. If not, then $P(z)\subset\Gamma$. In this case for each $u_\alpha - u_\beta\in U$ we have

$$z + u_1 + \cdots + u_s + u_\alpha - u_\beta = w + u_\alpha$$

where $w\in P(z)$, and as there are no error points in $P(z)$, we obtain $z+u_1+\cdots+u_s+u_\alpha+u_\beta\in\Gamma$. Since $z+u_1+\cdots+u_s\in P(z)\subset\Gamma$ and $z+u_1+\cdots+u_s$ is not an error point, we find that $z+u_1+\cdots+u_s$ is an internal point in Γ. Consider any internal point $z_0\in\Gamma$. If the point z_0+u_γ, $u_\gamma\in U$, is not internal, then either $y=z_0+u_\gamma$ is an error point or for some $u_\alpha, u_\beta\in U$ we have $z_0+u_\gamma+u_\alpha-u_\beta\in\partial\Gamma$. Thus $y-u_\beta=z_0+u_\gamma-u_\beta$ is an error point.

Now, taking the internal point $z+u_1+\cdots+u_s$, we shall start moving from it in the direction of vector u_1, then cyclically in the directions $u_2, u_3,..., u_s, u_1, u_2,...$, until we reach the first point y which is not internal. According to the above remark, either y is an error point or a point $x=y+u_\alpha$ is an error point for some $u_\alpha\in U$. We then assign this point, found via our algorithm, to z. Note that for any error point x, we can find all the adjacent points $z\in\Delta\Gamma$ to which it is assigned. For any given error point x, either z satisfies $P(z)\in x$, giving $|P(z)|=s^2-2s+2$ possibilities for z, or z is the first point of $\Delta\Gamma$ encountered when moving backward from x or $y=x+u_\alpha$, $u_\alpha\in U$, first in some direction $-u_\beta$ and then cyclically in the directions $-\pi_{-1}(u_\beta)$, $-\pi_{-2}(u_p)$, etc., giving $s(s+1)$ additional possibilities for z corresponding to choices of α and β. This fact proves the assertion of the lemma with $\alpha=1/(2s^2+2)$.

Remark 3. Now it is evident how to prove that the probability of having an infinite cluster containing some fixed point z_0 is equal to zero. Below we will give a detailed algorithm to prove it without going into all the technicalities, as they are just a restatement of those for the case of finite clusters.

Consider a sequence of cubes

$$J_n = \{(x_1, ..., x_d) \in Z^d \mid |x_i| \leqslant n, i = 1, ..., d\} \qquad \text{in } Z^d$$

and let $J_n^t = \bigcup_{n=0}^{t} \{-n \cdot u_1 + J_n\}$ be a parallepiped based on J_n and slanted in the direction of the vector $-u_1$. Let A_n be an event that there exists a cluster Γ containing the point z_0 intersecting one of the side faces of J_n^t. In keeping with the Borel–Cantelli lemma, to prove that the probability of an infinite cluster is equal to zero, it is enough to prove that there exists an $\varepsilon > 0$ such that for all t we can show that $\sum_{n=0}^{\infty} P(A_n) < \infty$. To do it, let us consider, for any cluster Γ intersecting one of the side faces of J_n^t, $n > t$, and containing the point z_0, a minimum $m \in \{0, 1, 2, ...\}$ such that a point $z_1 = z_0 + m \cdot u_1$ is an error point. Denote by $\Delta\Gamma_1$ the connected part of $\Delta\Gamma$ that belongs to J_n^t and contains this point z_1. In keeping with Lemma 2, the set $\Delta\Gamma_1$ must intersect one of the side faces of J_n^t, so its cardinality is greater than m. Thus to each cluster we assign a connected set $\Delta\Gamma_1 \bigcup_{i=1}^{m} (z_0 + i \cdot u_1)$. Repeating all arguments of Lemma 3, we obtain that a fixed proportion of points in this set are error points. This means that for some small $\varepsilon > 0$ for all $n > t$ the probability $P(A_n) < \beta^n$ for some $\beta < 1$. This implies the convergence of the considered series and finishes the proof of the statement. The details are left to the reader. Now we are ready to consider the time-space multistep cluster expansion for the probability $P(\sigma_{A_0}^-)$ of the configuration on an arbitrary finite set $A_0 \subset Z_t^{d+1}$ (we introduce "0" to show that we are at the starting point of our expansion). Taking $t = upA_0$, we consider all configurations $\underline{\sigma}$ in Z_t^{d+1} coinciding with $\sigma_{A_0}^-$ on A_0, and assign to each such configuration the cluster $\Gamma_0 = \Gamma_0(\underline{\sigma})$ with the kernel A_0 and the boundary $\partial\Gamma_0$. The expansion (3.2) for A_0 is then

$$P(\sigma_{A_0}^-) = \sum_{\Gamma_0} \sum_{\underline{\sigma}:\, \Gamma_0(\underline{\sigma}) = \Gamma_0} P(\underline{\sigma}) = \sum_{\Gamma_0} P(A_0^-, \Gamma_0^-, \partial\Gamma_0^+) \tag{3.5}$$

The probability on the right side of (3.5) can be rewritten as

$$\begin{aligned} P(A_0^-, \Gamma_0^-, \partial\Gamma_0^+) &= P(\Gamma_0^- \mid [A_0 \backslash \Gamma_0]^-, \partial\Gamma_0^+) \cdot P([A_0 \backslash \Gamma_0]^-, \partial\Gamma_0^+) \\ &= P(\Gamma_0^- \mid \partial\Gamma_0^+) \cdot P([A_0 \backslash \Gamma_0]^-, \partial\Gamma_0^+) \\ &= \varepsilon^{|\varepsilon(\Gamma_0)|} \cdot P([A_0 \backslash \Gamma_0]^-, \partial\Gamma_0^+) \end{aligned} \tag{3.6}$$

where the last equalities are the immediate consequences of the construction of the cluster. Next we rewrite probabilities on the right side of (3.6) as

$$P([A_0\backslash\Gamma_0]^-, \partial\Gamma_0^+) = (1-\varepsilon)^{|\sup \partial\Gamma_0|} \cdot P([A_0\backslash\Gamma_0]^-, [\partial\Gamma_0\backslash\sup \partial\Gamma_0]^+) \quad (3.7)$$

Using the fact that the probability of a plus configuration on some set may be rewritten as a sum of probabilities of minus configurations on the subsets of this set, we continue the expansion writing

$$\begin{aligned} &P([A_0\backslash\Gamma_0]^-, [\partial\Gamma_0\backslash\sup \partial\Gamma_0]^+) \\ &\quad = \sum_{W_0 \subset \partial\Gamma_0\backslash\sup \partial\Gamma_0 \cap Z^d \times \{1,2,\ldots\}} (-1)^{|W_0|} \cdot P([W_0 \cup A_0\backslash\Gamma_0]^-) \end{aligned} \quad (3.8)$$

Putting $A_1 = W_0 \cup A_0\backslash\Gamma_0$, it follows from (3.6)–(3.8) that

$$P(A_0^-) = \sum_{\Gamma_0} \sum_{W_0} (-1)^{|W_0|} \cdot (1-\varepsilon)^{|\sup \partial\Gamma_0|} \cdot \varepsilon^{|\varepsilon(\Gamma_0)|} \cdot P(A_1^-) \quad (3.9)$$

Noting that $up(A_1) < up(A_0)$, we now see how to continue the expansion (3.9). In exactly the same way that we constructed clusters with the kernel A_0, we now construct cluster Γ_1 with the kernel A_1 for each configuration in Z_{t-1}^{d+1} which agrees with $\sigma_{A_1}^-$, obtaining corresponding sets Γ_1, W_1, and $\partial\Gamma_1$. Thus defining the set $A_2 = W_1 \cup A_1\backslash\Gamma_1$, we rewrite the probability of the configuration on the set $A_1 \subset Z_{t-1}^{d+1}$ as the sum of the probabilities of the configurations on sets in Z_{t-2}^{d+1}. Since at every step we move down the t axis for no less than one unit of time, we continue the expansions of the form (3.6) for no more than t steps, indexing corresponding sets arising in the expansion by the number of the step. Thus we obtain

$$\begin{aligned} P(\sigma_{A_0}^-) &= \sum_{n=1}^{\infty} \sum_{\substack{A_0, \Gamma_0, W_0, \ldots, A_k, \Gamma_k, W_k: \\ |\Delta\Gamma_0| + \cdots + |\Delta\Gamma_k| = n}} (-1)^{|W_0| + \cdots + |W_k|} \\ &\qquad \times (1-\varepsilon)^{|\sup \partial\Gamma_0| + \cdots + |\sup \partial\Gamma_k|} \cdot \varepsilon^n \\ &= \sum_{n=1}^{\infty} c_n \varepsilon^n \end{aligned} \quad (3.10)$$

The sum is over all the chains of sets arising in the algorithm for the expansion under the condition $A_{k+1} = \varnothing$ for some $k \leqslant t$. Since $|\partial\Gamma_i| < d_0\, |\Delta\Gamma_i|$ for some constant $d_0 > 0$, the coefficient c_n in (3.10) is a polynomial in $(1-\varepsilon)$, and the degree of the polynomial is smaller than $d_0 \cdot n$. Using the estimate (3.3), we find that the coefficients of this polynomial are smaller in absolute value than C_1^n for some $C_1 > 0$. Applying the same construction for the minus configuration on the sets $T^{\tau_1}A_0$ and $T^{\tau_2}A_0$, we will obtain the expansions of these probabilities into the series

$$P(\sigma^-_{T^{\tau_1}A_0}) = \sum_n c_n^{\tau_1} \varepsilon^n \qquad \text{and} \qquad P(\sigma^-_{T^{\tau_2}A_0}) = \sum_n c_n^{\tau_2} \varepsilon^n$$

Looking at the expansion (3.10), we notice that the coefficients $c_n^{\tau_1}$ and $c_n^{\tau_2}$ are the sums of the same quantities over the congruent sets if the chain of the expansion is truncated before it reaches the $t=0$ level. That is why there exists a constant θ such that for $n < \theta \cdot \min(\tau_1, \tau_2)$ the coefficients $c_n^{\tau_1}$ and $c_n^{\tau_2}$ coincide. As these series converge exponentially for sufficiently small ε, we immediately obtain exponential convergence of the probabilities $P(\sigma^-_{T^{\tau}A_0})$ to some limit $\mu(\sigma^-_{A_0})$ for sufficiently small $\varepsilon > 0$ as τ tends to infinity. This fact together with the properties of the coefficients c_n^{τ} provides analyticity of $\mu(\sigma^-_{A_0})$ in ε in a neighborhood of zero.

We remark that the assertion of the theorem remains valid if the perturbation is of the following type:

$$f_{\underline{0}} = \begin{cases} 0, & \sigma_U = \sigma^-_U \\ \varepsilon(\sigma_U), & \sigma_U \neq \sigma^-_U \end{cases}$$

where $\varepsilon(\sigma_U)$ are analytic functions of ε in the neighborhood of zero, $|\varepsilon(\sigma_U)| \leqslant \text{const} \cdot \varepsilon$.

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